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Construction of a Complete Set of States in Relativistic Scattering Theory

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Abstract

The space of physical states in relativistic scattering theory is constructed, using a rigorous version of the Dirac formalism, where the Hilbert space structure is extended to a Gel'fand triple. This extension enables the construction of “a complete set of states”, the basic concept of the original Dirac formalism, also in the cases of unbounded operators and continuous spectra. We construct explicitly the Gel'fand triple and a complete set of “plane waves” – momentum eigenstates – using the group of space-time symmetries. This construction is used (in a separate article) to prove a generalization of the Coleman-Mandula theorem to higher dimension.

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I. Introduction

Scattering experiments are among the main sources of experimental information about the fundamental interactions in the sub-atomic range, therefore, any theory that is intended to describe these interactions, should provide predictions of scattering amplitudes. Much information about these amplitudes can be obtained using only very basic and well established assumptions as, for example, the fundamental postulates of quantum mechanics. This is the approach of *the theory of the S-matrix* (see, for example, ref. [1]). Results obtained with such an approach naturally have a wide range of validity and are applicable in any theory of fundamental interaction. They can be derived from the present standard model, but are also expected to remain valid when an improved fundamental description is found.

In the study of general properties of quantum mechanics it is natural to use Dirac's "bra" and "ket" formalism [2], because of its remarkable elegance and simplicity. The essence of its usefulness is the use of *a complete set of states* $\{< a|\}$ to form a representation of the unity operator

$$I \equiv \sum_a |a\rangle\langle a| \quad (\text{I.1})$$

which then can be used to represent various expression in terms of vector components $\langle a|\psi\rangle$ and matrix elements $\langle a'|A|a\rangle$ of operators. In scattering scenarios, a key role is played by the momentum operator and one would like to "diagonalize" it, *i.e.*, to take as a complete set of states, a set of its eigenvectors ("plane waves"). However, the spectrum of the momentum operator is continuous and the application of the Dirac formalism, in its original form, to such operators is not well defined, since they do not have a complete set of eigenvectors in the usual sense. This formalism is, therefore, unsuitable for rigorous analysis of scattering theory.

An improved version of the Dirac formalism was developed [3] [4] [5], using the theory of distributions [6]. In this formalism, the Hilbert space structure is extended to a *Gel'fand triple* $(\Phi, \mathcal{H}, \Phi')$ (called also *Rigged Hilbert Space*), where \mathcal{H} is the Hilbert space of states, Φ is a dense subspace of \mathcal{H} and Φ' is the *dual* of Φ – the space of continuous linear functionals on Φ . With an appropriate choice of Φ , a (generalized) complete set of eigenvectors of an operator with a continuous spectrum can be found among elements of Φ' and most of the elegance of the original formalism can be recovered.

In this work we use the improved formalism (following the presentation of Antoine [4] [5]), to construct a complete set of "plane waves" for relativistic scattering theory. In the spirit of the theory of the S-matrix, we use only very basic assumptions and state them explicitly, to make apparent the range of validity of the results. The construction relies on the symmetry of the theory under the group of (restricted) space-time transformations: translations, rotations and boosts. We consider an arbitrary dimension of space-time and assume the symmetry group to be (isomorphic to) $\mathcal{P}(r, s)$ – the inhomogeneous pseudo orthogonal group of signature (r, s) – with arbitrary r, s . Eventually we restrict to $s = 1$ and the complete space of "plane waves" is constructed for this case.

The structure of this article is as follows: In Section 2, the space of states is realized as a space of functions over momentum space; in Section 3 a Gel'fand triple and a "complete

set” of “plane waves” are constructed. Section 4 illustrates the use of the construction by rederiving some familiar relations and formulas (a further use of this construction is made in ref. [7]). Finally, in Section 5, we comment on the assumptions made and on possible extensions. Appendix A provides a concise description of the Dirac formalism used in this work. Appendix B summarizes some relevant properties of $\mathcal{P}(r, s)$.

II. The Space of states

In this Section we construct an explicit realization of the space of states as a space of functions over momentum space, using the space-time symmetry. In relativistic scattering theory, the space \mathcal{H}_s of physical states is a direct sum of (complex, separable) Hilbert spaces

$$\mathcal{H}_s = \bigoplus_{n=0}^{\infty} \mathcal{H}_s^{(n)} \quad (\text{II.1})$$

where $\mathcal{H}_s^{(n)}$ is the space of n -particle states (thus called “ n -particle space”) and is (isomorphic to) a closed subspace of the completed tensor product of n one-particle spaces:

$$\mathcal{H}_s^{(n)} \subset \mathcal{H}^{(n)} = \overline{\bigotimes_1^n \mathcal{H}^{(1)}}. \quad (\text{II.2})$$

The elements of $\mathcal{H}_s^{(n)}$ are those elements of $\mathcal{H}^{(n)}$ which have the right symmetry properties with respect to the exchange of identical particles.

The S-matrix S is assumed to be a unitary operator on \mathcal{H}_s .

II.1 The Poincaré Symmetry in $\mathcal{H}^{(1)}$

A symmetry transformation of the S-matrix is defined to be a unitary or antiunitary operator U in \mathcal{H}_s which satisfies:

1. $\mathcal{H}^{(1)}$ is U -invariant, *i.e.* U turns one-particle states into one-particle states;
2. U acts on many-particle states in accordance to their relation to the tensor product of one-particle states:

$$U(f_1 \otimes \cdots \otimes f_n) = (Uf_1) \otimes \cdots \otimes (Uf_n) \quad (\text{II.3})$$

(and thus, according to property 1, $\forall n$, $\mathcal{H}_s^{(n)}$ is U -invariant);

3. U commutes with S .

The invariance under (restricted) space-time transformations implies:

Assumption 2.1:

There exists a connected group \mathcal{P}'_0 of symmetries of S which is locally isomorphic to $\mathcal{P}(r, s)$.

The connectedness implies, among other things, that \mathcal{P}'_0 doesn't contain antiunitary elements. According to property 1 of the symmetries of S , $\mathcal{H}^{(1)}$ is \mathcal{P}'_0 -invariant and thus constitutes a representation space of \mathcal{P}'_0 . Any representation of \mathcal{P}'_0 is naturally also a representation of the universal covering group \mathcal{P} of \mathcal{P}'_0 which is (because of the local isomorphism) globally isomorphic to the universal covering group of the identity component (the largest connected subgroup) $\mathcal{P}_0 \equiv \mathcal{P}_0(r, s)$ of $\mathcal{P}(r, s)$ [8, p. 70] [9]. Wigner [10] and Bargmann [11] showed that this representation $U^{(1)}$ is in general a (strongly) continuous unitary projective representation (called also “ray representation” or “representation up to a phase”).

Assumptions:

2.2 $U^{(1)}$ is a true representation of \mathcal{P} (that is with no extra phase [12].

2.3 $U^{(1)}$ has only type I factors [8, p. 145] [13].

The second assumption means that $U^{(1)}$ is expressible in terms of irreducible representations and since these are identified with particle types (see the discussion in the introduction of [7]), this requirement is actually part of the physical interpretation..

As explained in appendix B.1, the most general such representation is of the following form (as usual, isomorphism between Hilbert spaces will be treated as equality):

$$U^{(1)} = \int_I d\rho(\alpha) U^\alpha \quad , \quad U^\alpha = (\chi_{p_{m_\alpha}} L^\alpha)(\mathcal{P}(p_{m_\alpha})) \upharpoonright \mathcal{P} \quad (\text{II.4})$$

and the representation space is:

$$\mathcal{H}^{(1)} = \int_I^\oplus d\rho(\alpha) \mathcal{H}_\alpha \quad , \quad \mathcal{H}_\alpha = \mathcal{L}_{\mu_{m_\alpha}}^2(\hat{\mathcal{T}}_{m_\alpha}, \mathcal{H}(L^\alpha)) \quad (\text{II.5})$$

where I is an index set, ρ is a measure on I (determined by $U^{(1)}$ up to equivalence) and for each $\alpha \in I$,

- i. $\hat{\mathcal{T}}_{m_\alpha}$ (the “ m_α -mass shell”) is an orbit of the “Lorentz group” $\mathcal{L}_0 = \mathcal{O}_0(r, s)$, in the “momentum space” $\hat{\mathcal{T}}$, and μ_{m_α} is the non-trivial \mathcal{L}_0 -invariant Radon measure on $\hat{\mathcal{T}}_{m_\alpha}$ (unique, up to a multiplicative constant); the non-triviality and the \mathcal{L} -invariance of μ_{m_α} imply that it is non-degenerate (it does not vanish on open sets);
- ii. p_{m_α} is the representative of $\hat{\mathcal{T}}_{m_\alpha}$ (as chosen in appendix B, *e.g.*, for the time-like orbits of $\mathcal{O}(r, 1)$, $p = (p, 0, \dots, 0)$, which in 4 dimensions corresponds to the rest frame);

- iii. L^α is an irreducible continuous unitary representation of the little group $\mathcal{L}(p_{m_\alpha})$ of p_{m_α} (where \mathcal{L} is the universal covering group of \mathcal{L}_0) in the (complex separable) Hilbert space $\mathcal{H}(L^\alpha)$;
- iv. \mathcal{H}_α is the space of μ_{m_α} -square-integrable functions on $\hat{\mathcal{T}}_{m_\alpha}$, taking values in $\mathcal{H}(L^\alpha)$;
- v. U^α is the irreducible (continuous and unitary) representation of \mathcal{P} in \mathcal{H}_α , induced by L^α :

$$[U^\alpha(\Lambda, a)f](p) = e^{ip \cdot a} L^\alpha(\Delta(\Lambda, p))f(\Lambda^{-1}p) \quad (\text{II.6})$$

where

$$\Delta(\Lambda, p) = \Lambda_p^{-1} \Lambda \Lambda_{\Lambda^{-1}p}$$

and $\forall p \in \hat{\mathcal{T}}_{m_\alpha}$, Λ_p is in \mathcal{L} and satisfies $\Lambda_p p_{m_\alpha} = p$; thus $\Delta(\Lambda, p) \in \mathcal{L}(p_{m_\alpha})$.

We want to treat the elements of $\mathcal{H}^{(1)}$ as vector-valued functions on $\hat{\mathcal{T}}$. For this, we assume that the order of the $d\rho(\alpha)$ integration can be arranged to be

$$\int_I d\rho(\alpha) \dots = \int_{\mathcal{M}} d\hat{\mu}(m) \int_{I(m)} d\rho(\alpha) \dots \quad (\text{II.7})$$

where $\hat{\mu}$ is a measure on a set of orbits $\{\hat{\mathcal{T}}_m | m \in \mathcal{M}\}$ and for each $m \in \mathcal{M}$, $I(m) := \{\alpha \in I | m_\alpha = m\}$ is the set of indices of all irreducible components of $U^{(1)}$ with the “mass” m . (This assumption should be satisfied if I is not pathological.) In this case we have (compare to (II.5)):

$$\mathcal{H}^{(1)} = \int_{\mathcal{M}}^{\oplus} d\hat{\mu}(m) \int_{\hat{\mathcal{T}}_m}^{\oplus} d\mu_m(p) \mathcal{H}(m) \quad , \quad \mathcal{H}(m) := \int_{I(m)}^{\oplus} d\rho(\alpha) \mathcal{H}(L^\alpha) \quad (\text{II.8})$$

and $U^{(1)}$ gets the form (compare to (II.6)): for $p \in \hat{\mathcal{T}}_m$,

$$[U^{(1)}(\Lambda, a)f](p) = e^{ip \cdot a} L^{(p)}(\Delta(\Lambda, p))f(\Lambda^{-1}p) \quad (\text{II.9})$$

where $L^{(p)} = \int_{I(m)} d\rho(\alpha) L^\alpha$ is the (reducible) unitary representation of $\mathcal{L}(p_m)$ in $\mathcal{H}(m)$.

The momentum support of the elements of $\mathcal{H}^{(1)}$ is restricted to $\hat{\mathcal{T}}_F := \bigcup_{m \in \mathcal{M}} \hat{\mathcal{T}}_m$. This will be called “the (one particle) physical region in $\hat{\mathcal{T}}$ ” and it is the spectrum of the momentum operator in $\mathcal{H}^{(1)}$.

II.2 Representing \mathcal{H}_s as a Function Space

The next step is to extend $U^{(1)}$ from \mathcal{P} to $\mathcal{A}_{\mathcal{P}}$, the Lie algebra of \mathcal{P} . This is done by identifying the elements of \mathcal{P} and $\mathcal{U}_{\mathcal{P}}$, the universal enveloping algebra of $\mathcal{A}_{\mathcal{P}}$, as distributions on \mathcal{P} with compact support, and then defining a representation of $\mathcal{E}'(\mathcal{P})$, the space of all such distributions. In this procedure, we follow Antoine [5] and it is described in appendix B.2 (refer also to appendix A for notation and terminology). The results can be summarized as follows:

If

- C1.** $(\Psi_\alpha, \mathcal{H}(L^\alpha), \Psi'_\alpha)$ is a Gel'fand triple:
 - (a) Ψ_α is a complete nuclear space, embedded in $\mathcal{H}(L^\alpha)$ densely and continuously,
 - (b) Ψ'_α is the strong dual of Ψ_α ,
- C2.** the restriction of L^α to Ψ_α is a smooth representation of $\mathcal{L}(p_{m_\alpha})$ by continuous operators in Ψ_α :
 - (a) Ψ_α is L^α -invariant,
 - (b) for each $\Delta \in \mathcal{L}(p_{m_\alpha})$, $L^\alpha(\Delta)$ is a continuous operator in Ψ_α ,
 - (c) for each $\psi \in \Psi_\alpha$, the function $\Delta \mapsto L^\alpha(\Delta)\psi$ is a smooth function from $\mathcal{L}(p_{m_\alpha})$ to $\mathcal{H}(L^\alpha)$,
- C3.** the map $p \mapsto \Lambda_p$ is smooth on $\hat{\mathcal{T}}_{m_\alpha}$,

then

- R1.** the triple $(\Phi_\alpha, \mathcal{H}_\alpha, \Phi'_\alpha)$, where Φ_α is defined by

$$\Phi_\alpha := \mathcal{D}(\hat{\mathcal{T}}_{m_\alpha}; \Psi_\alpha), \quad (\text{II.10})$$

is a Gel'fand triple, having properties (1a) and (1b) of $(\Psi_\alpha, \mathcal{H}(L^\alpha), \Psi'_\alpha)$,

- R2.** the restriction of U^α to Φ_α is a smooth representation of \mathcal{P} by continuous operators in Φ_α (*i.e.* U^α has the properties (2a), (2b), and (2c) of L^α),
- R3.** in Φ_α , U^α is naturally extended to a continuous *-representation of $\mathcal{U}_{\mathcal{P}}$ (actually, of all of $\mathcal{E}'(\mathcal{P})$) by continuous operators (the elements of $\mathcal{U}_{\mathcal{P}}$ being represented by differential operators on $\hat{\mathcal{T}}_{m_\alpha}$):

$$U^\alpha(A)U^\alpha(B) = U^\alpha(AB) \quad , \quad \forall A, B \in \mathcal{U}_{\mathcal{P}}$$

$$U^\alpha(A^\dagger) \subset [U^\alpha(A)]^* \quad , \quad \forall A \in \mathcal{U}_{\mathcal{P}}$$

so if A is symmetric ($A^\dagger = A$) then U^α is Hermitian.

- R4.** if $A \in \mathcal{U}_{\mathcal{P}}$ satisfies the Nelson-Stinespring criterion (A is elliptic – when considered as a differential operator on $\mathcal{E}(\mathcal{P})$ – or there exists $B \in \mathcal{U}_{\mathcal{P}}$ symmetric and elliptic for which $[U^\alpha(A), U^\alpha(B^\dagger B)] = 0$ on Φ_α) then

$$\overline{U^\alpha(A^\dagger)} = [U^\alpha(A)]^*$$

($\overline{U^\alpha(A^\dagger)}$ is the closure of $U^\alpha(A^\dagger)$ in \mathcal{H}_α) so if A is symmetric then $U^\alpha(A)$ is essentially self adjoint [14],

- R5.** if $A, B \in \mathcal{U}_{\mathcal{P}}$ are represented by essentially self adjoint operators $U^\alpha(A)$, $U^\alpha(B)$ then:

$$\overline{U^\alpha(A)} \text{ and } \overline{U^\alpha(B)} \text{ strongly commute} \iff [A, B] = 0. \quad (\text{II.11})$$

In this situation (*i.e.* under the conditions C1 and C2) it is possible to represent $\mathcal{H}^{(1)}$ as a space of complex-valued functions. Let $P = P^\mu$ be the “momentum” operator (the generator of translations: $iP^\mu = \partial/\partial a_\mu|_{a=0}$). From eq. (II.6) one gets that

$$\forall \varphi \in \Phi_\alpha \quad , \quad [U^\alpha(P^\mu)\varphi](p) = p^\mu \varphi(p)$$

so P is diagonal in Φ_α (it also demonstrates all the results R3, R4 and R5). Using results R4 and R5 and the irreducibility of U^α , we choose a (finite) abelian set $J = \{J_i\}$ of symmetric elements of \mathcal{U}_P such that for each $\alpha \in I$, $\{\overline{U^\alpha(P)}, \overline{U^\alpha(J)}\}$ is a complete system of strongly commuting self adjoint operators and we diagonalize it, using von Neumann’s complete spectral theorem [15, p. 54]. This diagonalization will only affect $\mathcal{H}(L^\alpha)$ (which is, in a generalized sense, an eigenspace of P) so we get

$$\mathcal{H}(L^\alpha) \simeq \mathcal{L}_{\nu_\alpha}^2(\sigma_\alpha) \quad (\text{II.12})$$

where σ_α is the spectrum of $\overline{U^\alpha(J)}$ and ν_α is a spectral measure on σ_α . Therefore:

$$\mathcal{H}_\alpha = \mathcal{L}_{\mu_{m_\alpha}}^2(\hat{\mathcal{T}}_{m_\alpha}; \mathcal{H}(L^\alpha)) \simeq \mathcal{L}_{\mu_{m_\alpha} \times \nu_\alpha}^2(\hat{\mathcal{T}}_{m_\alpha} \times \sigma_\alpha) \quad (\text{II.13})$$

and for each $\varphi \in \Phi_\alpha (\subset \mathcal{H}_\alpha)$, $p \in \hat{\mathcal{T}}_{m_\alpha}$, $\lambda \in \sigma_\alpha$:

$$[U^\alpha(P^\mu)\varphi](p, \lambda) = p^\mu \varphi(p, \lambda) \quad (\text{II.14})$$

$$[U^\alpha(J_i)\varphi](p, \lambda) = \lambda_i \varphi(p, \lambda). \quad (\text{II.15})$$

Turning now to $\mathcal{H}^{(1)}$, we have

$$\mathcal{H}(m) = \int_{I(m)}^\oplus d\rho(\alpha) \mathcal{H}(L^\alpha) = \mathcal{L}_{\rho_m}^2(\Omega(m)) \quad (\text{II.16})$$

where

$$\Omega(m) := \{[\alpha\lambda] | \alpha \in I(m), \lambda \in \sigma_\alpha\}$$

and

$$\int_{\Omega(m)} d\rho_m(\alpha, \lambda) \dots := \int_{I(m)} d\rho(\alpha) \int_{\sigma_\alpha} d\nu_\alpha(\lambda) \dots$$

thus (compare to (II.8)):

$$\mathcal{H}^{(1)} = \mathcal{L}_\mu^2(\Omega) \quad (\text{II.17})$$

where

$$\begin{aligned} \Omega &:= \{(p, \lambda, \alpha) | m \in \mathcal{M}, p \in \hat{\mathcal{T}}_m, [\alpha\lambda] \in \Omega(m)\} \\ &= \{(p, \lambda, \alpha) | \alpha \in I, p \in \hat{\mathcal{T}}_{m_\alpha}, \lambda \in \sigma_\alpha\} \end{aligned}$$

and

$$\int_\Omega d\mu(p, \lambda, \alpha) \dots = \int_{\mathcal{M}} d\hat{\mu}(m) \int_{\hat{\mathcal{T}}_m} d\mu_m(p) \int_{I(m)} d\rho(\alpha) \int_{\sigma_\alpha} d\nu_\alpha(\lambda) \dots$$

The n -particle space is, according to (II.2) [16]

$$\mathcal{H}^{(n)} = \overline{\bigotimes_1^n \mathcal{L}_\mu^2(\Omega)} = \mathcal{L}_{\mu^n}^2(\Omega^n) \quad (\text{II.18})$$

where

$$\Omega^n := \Omega \times \cdots \times \Omega \text{ (} n \text{ factors)} \quad (\text{II.19})$$

and the measure μ^n is defined by

$$\int_{\Omega^n} d\mu^n((p, \lambda, \alpha)^n) \dots := \int_{\Omega} d\mu(p_1, \lambda_1, \alpha_1) \dots \int_{\Omega} d\mu(p_n, \lambda_n, \alpha_n) \dots \quad (\text{II.20})$$

III. The Complete Set of Plane Waves

At this point we introduce further assumptions about the signature of space-time and the spectrum of particle types:

Assumptions:

3.1 $\mathcal{L}_0 = \mathcal{O}_0(r, 1)$, $r \geq 3$ and a momentum p in the physical region \hat{T}_F is time-like ($p_\mu p^\mu > 0$) and in the forward light cone ($E > 0$) (thus m can be recognized as a generalized mass $m = \sqrt{p_\mu p^\mu}$; see Appendix B for notation).

3.2 I is countable (thus ρ is a purely atomic measure) and $\{m_\alpha\}_{\alpha \in I}$ is a non-decreasing sequence.

From now on, we also assume that the map $p \mapsto \Lambda_p$ was chosen to be smooth and $\Lambda_{p_m} = 1$, $\forall m \in \mathcal{M}$.

III.1 The One-Particle Space

Assumption 3.1 means that $\mathcal{L}_0(p_m) = \mathcal{O}_0(r)$ which is a compact group. Since $r \neq 2$, it is at most doubly covered by $\mathcal{L}(p_m) = \overline{\mathcal{O}_0(r)}$, therefore $\mathcal{L}(p_m)$ is also compact, so all its irreducible representations are finite dimensional: $\forall \alpha \in I$, $\dim \mathcal{H}(L^\alpha) < \infty$. This implies that:

(1a) $\mathcal{H}(L^\alpha)$ is a nuclear space [17, p. 520];

(1b) since L^α is continuous, it is smooth (even analytic [8, p. 322]);

thus all the requirements for the extension of U^α to $\mathcal{U}_{\mathcal{P}}$ are satisfied with $\Psi_\alpha = \mathcal{H}(L^\alpha)$.

Assumption 3.1 also implies that all the ray representations of \mathcal{P} are (equivalent to) true representations [12], so assumption 2.2 (in Section II.1) is satisfied.

Assumption 3.2 means that:

(2a) \mathcal{M} (the mass spectrum) is a countable set of discrete points in $(0, \infty)$ (with no accumulation points). Thus

$$\mathcal{H}^{(1)} = \bigoplus_{m \in \mathcal{M}} \mathcal{L}_{\mu_m}^2(\hat{\mathcal{T}}_m, \mathcal{H}(m)); \quad (\text{III.1})$$

(2b) For each $m \in \mathcal{M}$, $I(m)$ (the set of indices of all irreducible components of $U^{(1)}$ with the “mass” m) is finite:

$$\mathcal{H}(m) = \bigoplus_{\alpha \in I(m)} \mathcal{H}(L^\alpha), \quad L^{(p)} = \bigoplus_{\alpha \in I(m)} L^\alpha, \quad \text{where } m = \sqrt{p_\mu p^\mu} \quad (\text{III.2})$$

(2c) $U^{(1)}$ is a direct sum of irreducible representations:

$$\mathcal{H}^{(1)} = \bigoplus_{\alpha \in I} \mathcal{H}_\alpha, \quad U^{(1)} = \bigoplus_{\alpha \in I} U^\alpha \quad (\text{III.3})$$

Combining both assumptions, we get:

(3a) $\mathcal{H}(m)$ is finite-dimensional and since it is (identified with) a space of functions, we have:

$$\mathcal{H}(m) = \mathbf{C}^{N(m)} \quad (N(m) := \dim \mathcal{H}(m)) \quad (\text{III.4})$$

thus operators in $\mathcal{H}(m)$ (e.g. $L^{(p)}$) are matrices.

(3b) Ω (defined in eq. (II.17)) is a countable union of orbits. Since each orbit is a separable smooth manifold, so is Ω .

The measure μ on Ω takes the form:

$$\begin{aligned} \int_{\Omega} d\mu(p, \lambda, \alpha) \dots &= \sum_{\alpha \in I} \int_{\hat{\mathcal{T}}_{m_\alpha}} d\mu_{m_\alpha}(p) \sum_{\lambda \in \sigma_\alpha} \dots \\ &= \sum_{m \in \mathcal{M}} \int_{\hat{\mathcal{T}}_m} d\mu_m(p) \sum_{[\alpha \lambda] \in \Omega(m)} \dots \end{aligned} \quad (\text{III.5})$$

(compare to eq. (II.17)) and since on each orbit $\hat{\mathcal{T}}_m$, μ_m is a non degenerate (\mathcal{L} -invariant) Radon measure, so is μ on Ω .

Writing everything as a function on the momentum space, it is frequently convenient to write $\mathcal{H}(p)$, $N(p)$, $\Omega(p)$ and $I(p)$ instead of $\mathcal{H}(m)$ etc., where $m = \sqrt{p_\mu p^\mu}$.

Now we are able to identify the structure of the space of states required by the Dirac formalism (see appendix A.2). Recalling that $\mathcal{H}^{(1)} = \mathcal{L}_\mu^2(\Omega)$, We define:

$$\Phi^{(1)} := \mathcal{D}(\Omega). \quad (\text{III.6})$$

Since I is countable, it can be chosen to be a set of natural numbers, so the operator $\hat{\alpha}$ defined by

$$[\hat{\alpha}f](p, \lambda, \alpha) := \alpha f(p, \lambda, \alpha), \quad \forall f \in \mathcal{H}^{(1)} \quad (\text{III.7})$$

is self adjoint and diagonal. Recognizing that $\mathcal{L}_\mu^2(\Omega)$ is the spectral decomposition of $\mathcal{H}^{(1)}$ with respect to the system $\{\hat{\alpha}, \overline{U^{(1)}(P^\mu)}, \overline{U^{(1)}(J_i)}\}$, one obtains that this is a complete system of strongly commuting self adjoint operators. Next, we recognize that

$$\Phi^{(1)} = \sum_{\alpha \in I} \Phi_\alpha \quad (\text{III.8})$$

(a topological direct sum of locally convex spaces: the set of *finite* sums of elements of $\{\Phi_\alpha\}_{\alpha \in I}$ [17, p. 515]) so (according to result R3 of Section II.2) all the above operators, when restricted to $\Phi^{(1)}$ are continuous operators in $\Phi^{(1)}$. Thus $(\Phi^{(1)}, \mathcal{H}^{(1)}, \Phi^{(1)'})$ is a Gel'fand triple with all the properties required, so all the results described in appendix A apply here. In particular, the set $\{< p, \lambda, \alpha | \}_{(p, \lambda, \alpha) \in \Omega}$, defined (μ -almost everywhere) by

$$< p, \lambda, \alpha | \varphi \rangle := \varphi(p, \lambda, \alpha), \quad \forall \varphi \in \Phi^{(1)} \quad (\text{III.9})$$

is a “complete orthonormal” system of eigenbras of $\{\hat{\alpha}', \overline{U^{(1)}(P^\mu)'}, \overline{U^{(1)}(J_i)'}\}$ (in the sense explained in appendix A.4).

In the following, to simplify notation, the indices $[\alpha \lambda]$ will be usually omitted, making the summations implicit (“matrix multiplication”); $< p, \lambda, \alpha |$ and $d\mu(p, \lambda, \alpha)$ are abbreviated by $< p |$ and $d\mu(p)$.

III.2 The Total Space

The n -particle space is (see (II.18))

$$\mathcal{H}^{(n)} = \mathcal{L}_{\mu^n}^2(\Omega^n)$$

where Ω^n is a separable smooth manifold and μ^n is a non degenerate Radon measure (see result 3b in the previous subsection). This suggests that the natural choice for $\Phi^{(n)}$ is

$$\Phi^{(n)} := \mathcal{D}(\Omega^n), \quad (\text{III.10})$$

obtaining a Gel'fand triple $(\Phi^{(n)}, \mathcal{H}^{(n)}, \Phi^{(n)'})$.

For each $p^n = (p_1, \dots, p_n) \in \Omega^n$ we define:

$$< p^n | := < p_1 | \otimes \dots \otimes < p_n | \quad (\text{III.11})$$

and since $\bigotimes_1^n \mathcal{D}'(\Omega)$ is a (dense) subspace of $\mathcal{D}'(\Omega^n)$ [17, p. 417], we get

$$< p^n | \in \Phi^{(n)'} \quad (\text{III.12})$$

with the action (for μ^n -almost all $p^n \in \Omega^n$)

$$< p^n | \varphi \rangle = \varphi(p^n), \quad \forall \varphi \in \Phi^{(n)}. \quad (\text{III.13})$$

For $\varphi, \psi \in \otimes_1^n \mathcal{D}(\Omega)$, we have

$$(\psi|\varphi) = (\psi, \varphi)_{\mathcal{H}^{(n)}} = \prod_{i=1}^n (\psi_i, \varphi_i)_{\mathcal{H}^{(1)}} \quad (\text{III.14})$$

$$\begin{aligned} &= \prod_{i=1}^n \int_{\Omega} d\mu(p_i) \overline{\psi_i(p_i)} \varphi_i(p_i) \\ &= \int_{\Omega^n} d\mu^n(p^n) (\psi|p^n \rangle \langle p^n|\varphi). \end{aligned} \quad (\text{III.15})$$

This is exactly the Parseval equality (compare to eq. (A.12)) which means that $\{ \langle p^n | \}$ is a complete orthonormal system of bras in the sense that the operator

$$I^n := \int_{\Omega^n} d\mu^n(p^n) |p^n \rangle \langle p^n| \quad (\text{III.16})$$

is the embedding of $\Phi^{(n)}$ into $\Phi^{(n)'}$ and plays the role of the identity operator in the Dirac formalism [18].

Next, we define

$$\mathcal{H} := \bigoplus_{n=1}^{\infty} \mathcal{H}^{(n)}, \quad \Phi := \sum_{n=1}^{\infty} \Phi^{(n)} \quad (\text{III.17})$$

obtaining a Gel'fand triple $(\Phi, \mathcal{H}, \Phi')$. Since $\Phi^{(n)}$ is a closed subspace of Φ , $\Phi^{(n)'}$ is a quotient space of Φ' and can be naturally identified as a (closed) subspace of Φ' , by defining

$$\langle \Phi^{(m)'} | \Phi^{(n)} \rangle \propto \delta_{mn}. \quad (\text{III.18})$$

Therefore we get

$$\Phi' = \prod_{n=1}^{\infty} \Phi^{(n)'}, \quad (\text{III.19})$$

(the set of *arbitrary infinite* sums) and the embedding of Φ into Φ' is

$$I := \sum_{n=1}^{\infty} I^n = \sum_{n=1}^{\infty} \int_{\Omega^n} d\mu^n(p^n) |p^n \rangle \langle p^n|. \quad (\text{III.20})$$

Finally, we define

$$\Phi_s := \Phi \cap \mathcal{H}_s, \quad \Phi_s^{(n)} := \Phi^{(n)} \cap \mathcal{H}_s^{(n)} \quad (\text{III.21})$$

obtaining the Gel'fand triples $(\Phi_s, \mathcal{H}_s, \Phi_s')$ and $(\forall n) (\Phi_s^{(n)}, \mathcal{H}_s^{(n)}, \Phi_s^{(n)'})$, equipped with the complete systems of bras defined above.

III.3 Matrix Elements

Let $A \in L^\times(\Phi; \Phi')$. As stated in appendix A.3, this space contains all the spaces $L(\Phi), L(\mathcal{H}), L(\Phi; \mathcal{H}), L^\times(\mathcal{H}; \Phi')$, and $L(\Phi')$ where “ L ” denotes “linear” and “ L^\times ” – “antilinear”. Moreover, Φ_s is a closed subspace of Φ so if $A \in L^\times(\Phi_s; \Phi_s')$, it can always be

extended continuously to all of Φ by defining $Af := 0$ for each $f \in \mathcal{H}$ which is orthogonal to Φ_s . This means that $L^\times(\Phi_s; \Phi'_s)$, and all the corresponding spaces of continuous mappings, are also embedded naturally in $L^\times(\Phi; \Phi')$.

To define the matrix elements of A , we decompose it to “elements” acting between states of definite number of particles. Let τ^n be the natural (continuous isomorphic) embedding of $\Phi^{(n)}$ in Φ , and $\tau^{n'}$, the dual mapping (this is the natural projection of Φ' on $\Phi^{(n)'}).$ We define

$$A^{(m,n)} := \tau^{m'} A \tau^n \in L^\times(\Phi^{(n)}; \Phi^{(m)'}) \quad (\text{III.22})$$

obtaining, for each $\varphi, \psi \in \Phi$

$$(\psi|A\varphi > = \sum_{m,n} (\psi^m|A^{(m,n)}\varphi^n > \quad (\text{III.23})$$

(with the notation $\varphi = \sum_n \varphi^n$, $\varphi^n \in \Phi^{(n)}$ and the same for ψ).

Now we can apply the kernel theorem (see appendix A.4), obtaining

$$(\psi^m|A^{(m,n)}\varphi^n > = \int_{\Omega^m \times \Omega^n} d\mu^m(q^m) d\mu^n(p^n) \overline{\psi^m(q^m)} A^{(m,n)}(q^m, p^n) \varphi^n(p^n) \quad (\text{III.24})$$

where $< A^{(m,n)} > \in \mathcal{D}'(\Omega^m \times \Omega^n)$ is the kernel that corresponds to $A^{(m,n)}$. Denoting

$$< q^m|A|p^n > := A^{(m,n)}(q^m, p^n) \quad (\text{III.25})$$

we obtain (combining (III.23) and (III.24))

$$(\psi|A\varphi > = \sum_{m,n} \int_{\Omega^m \times \Omega^n} d\mu^m(q^m) d\mu^n(p^n) (\psi|q^m > < q^m|A|p^n > < p^n|\varphi) \quad (\text{III.26})$$

so we see that , as in the general formalism, the expression (III.20) for I can be formally inserted between the factors of $(\psi|A|\varphi >$.

IV. Applications

In this Section we illustrate the formalism derived above by rederiving familiar relations and formulas using the new language. The resemblance to the original Dirac formalism is apparent but unlike the original formalism, in the present formulation all the expressions have a well defined meaning.

IV.1 The Representation $U^{(1)}$

From eq. (II.9) we obtain

$$< p|U^{(1)}(\Lambda, a)|\varphi > = [U^{(1)}(\Lambda, a)\varphi](p) = \quad (\text{IV.1})$$

$$= e^{ip \cdot a} L^{(p)}(\Delta(\Lambda, p)) < \Lambda^{-1}p|\varphi > \quad (\text{IV.2})$$

so the action of $U^{(1)}$ on the base vectors is

$$\langle p|U^{(1)}(\Lambda, a) = e^{ip \cdot a} L^{(p)}(\Delta(\Lambda, p)) \langle \Lambda^{-1}p|. \quad (\text{IV.3})$$

(here $\langle p|$ is considered as a column vector — because when acting on $|\varphi\rangle$, it produces a column vector in $\mathcal{H}(p)$ — thus $|p\rangle$ is recognized as a row vector).

It is more customary to write the expression for $U^{(1)}(\Lambda, a)|p\rangle$ and this is equal to $(\langle p|U^{(1)}(\Lambda, a)^*)^\dagger$, where “ \dagger ” denotes the matrix conjugation in $\mathcal{H}(p)$ (and not the adjoint defined in appendix A.4). $U^{(1)}(\Lambda, a)$ is unitary and $(\Lambda, a)^{-1} = (\Lambda^{-1}, -\Lambda^{-1}a)$ (see appendix B), so

$$\begin{aligned} U^{(1)}(\Lambda, a)|p\rangle &= [\langle p|U^{(1)}(\Lambda^{-1}, -\Lambda^{-1}a)]^\dagger = \\ &= [e^{-ip \cdot \Lambda^{-1}a} L^{(p)}(\Delta(\Lambda^{-1}, p)) \langle \Lambda p|]^\dagger = \\ &= |\Lambda p\rangle e^{ip \cdot \Lambda^{-1}a} L^{(p)}(\Delta(\Lambda^{-1}, p))^\dagger \end{aligned}$$

$L^{(p)}$ is a unitary representation and

$$\Delta(\Lambda^{-1}, p)^{-1} = (\Lambda_p^{-1} \Lambda^{-1} \Lambda_{\Lambda p})^{-1} = \Lambda_{\Lambda p}^{-1} \Lambda \Lambda_p = \Delta(\Lambda, \Lambda p)$$

so (since also $p \cdot \Lambda^{-1}a = a \cdot \Lambda p$)

$$U^{(1)}(\Lambda, a)|p\rangle = |\Lambda p\rangle e^{ia \cdot \Lambda p} L^{(p)}(\Delta(\Lambda, \Lambda p)) \quad (\text{IV.4})$$

and explicitly, with components:

$$U^{(1)}(\Lambda, a)|p, \lambda, \alpha\rangle = e^{ia \cdot \Lambda p} L^\alpha(\Delta(\Lambda, \Lambda p))_{\lambda' \lambda} |\Lambda p, \lambda', \alpha\rangle. \quad (\text{IV.5})$$

This also implies that the matrix elements of $U^{(1)}$ are

$$\langle p'|U^{(1)}(\Lambda, a)|p\rangle = e^{ia \cdot \Lambda p} L^{(p)}(\Delta(\Lambda, \Lambda p))_{\mu} \delta_\mu(p' - \Lambda p) \quad (\text{IV.6})$$

or, in components:

$$\langle p', \lambda', \alpha'|U^{(1)}(\Lambda, a)|p, \lambda, \alpha\rangle = e^{ia \cdot \Lambda p} \delta_{\alpha' \alpha} L^\alpha(\Delta(\Lambda, \Lambda p))_{\lambda' \lambda} \delta_\mu(p' - \Lambda p) \quad (\text{IV.7})$$

and indeed, one gets

$$\langle p|U^{(1)}(\Lambda, a)|\varphi\rangle = \int_\Omega d\mu(p) \langle p|U^{(1)}(\Lambda, a)|p'\rangle \langle p'|\varphi\rangle. \quad (\text{IV.8})$$

IV.2 Generators of Symmetry

Let $g(t)$ be a one-parameter symmetry group of S . As such, it is represented in $\mathcal{H}^{(1)}$ by a unitary representation $U^{(1)}$. The generator A_g of $U^{(1)}(g(t))$ is defined by

$$(\psi|A_g\varphi\rangle := \frac{1}{i} \frac{d}{dt} (\psi, U^{(1)}(g(t))\varphi)|_{t=0}, \quad \forall \varphi, \psi \in \Phi^{(1)} \quad (\text{IV.9})$$

and it is assumed to be an element of $L^\times(\Phi^{(1)}; \Phi^{(1)'})$. (this means, in particular, that the derivative exists for each φ, ψ and the function thus obtained is continuous in ψ and φ) [19]. If $g(t)$ is a subgroup of \mathcal{P} , this assumption is certainly satisfied: $\Phi^{(1)}$ was intentionally constructed to make a generator of such a group a continuous operator in $\Phi^{(1)}$.

The unitarity of $U^{(1)}$:

$$U^{(1)}(g(t))^* = U^{(1)}(g(t)^{-1}) = U^{(1)}(g(-t)) \quad (\text{IV.10})$$

implies that A_g is self adjoint (as an element of $L^\times(\Phi^{(1)}; \Phi^{(1)'})$: $A^\dagger = A$):

$$\begin{aligned} (\psi|A_g^\dagger\varphi> &= \overline{(\varphi|A_g\psi>} = \overline{\frac{1}{i}\frac{d}{dt}(\varphi, U^{(1)}(g(t))\psi)|_{t=0}} = \\ &= -\frac{1}{i}\frac{d}{dt}(\psi, U^{(1)}(g(t))^*\varphi)|_{t=0} = \\ &= -\frac{1}{i}\frac{d}{dt}(\psi, U^{(1)}(g(-t))\varphi)|_{t=0} = \\ &= \frac{1}{i}\frac{d}{dt}(\psi, U^{(1)}(g(t))\varphi)|_{t=0} = (\psi|A_g\varphi> \end{aligned} \quad (\text{IV.11})$$

Next we turn to multi-particle states. $A_g^{(m,n)}$ is defined naturally by:

$$(\psi^m|A_g^{(m,n)}\varphi^n> := \frac{1}{i}\frac{d}{dt}(\psi^m, U(g(t))\varphi^n)|_{t=0}, \quad \forall \psi^m \in \Phi^{(m)}, \varphi \in \Phi^{(n)}. \quad (\text{IV.12})$$

Since U doesn't change the number of particles, neither does A_g , so

$$A_g^{(m,n)} = 0, \quad \forall m \neq n. \quad (\text{IV.13})$$

For $\psi = \bigotimes_1^n \psi_i$, $\varphi = \bigotimes_1^n \varphi_i$ in $\bigotimes_1^n \Phi^{(1)}$ we have

$$\begin{aligned} (\psi|A_g^{(n)}\varphi> &= \frac{1}{i}\frac{d}{dt}[\prod_{i=1}^n (\psi_i, U^{(1)}(g(t))\varphi_i)]|_{t=0} \\ &= \sum_{i=1}^n (\psi_i|A_g\varphi_i> \prod_{j \neq i} (\psi_j, \varphi_j) \end{aligned} \quad (\text{IV.14})$$

so at least between elements of $\bigotimes_1^n \Phi^{(1)}$:

$$A_g^{(n)} = (A_g \otimes I \otimes \cdots \otimes I) + (I \otimes A_g \otimes \cdots \otimes I) + \cdots + (I \otimes I \otimes \cdots \otimes A_g) \quad (\text{IV.15})$$

(to be extended to arbitrary elements of $\Phi^{(n)}$, $A_g^{(n)}$ must be continuous).

Finally, $g(t)$, being a group of symmetries of the S-matrix S , satisfies $[U(g(t)), S] = 0$. This implies that for each $\varphi, \psi \in \Phi$,

$$(S^*\psi, U(g(t))\varphi) = \overline{(S\varphi, U(g(-t))\psi)}. \quad (\text{IV.16})$$

Differentiating, one gets (when $S^*\psi, S\varphi \in \Phi$)

$$(S^*\psi|A_g\varphi> = <A_g\psi|S\varphi). \quad (\text{IV.17})$$

In particular, if S and A_g are operators in Φ then

$$[A_g, S] = 0 \text{ in } \Phi. \quad (\text{IV.18})$$

Also, if A_g is a *continuous operator in* Φ (e.g. a generator of \mathcal{P}) then (IV.18) holds, with the commutators defined to be

$$[A_g, S] = A'_g S - S A_g \quad (\text{IV.19})$$

where A'_g is the dual of A_g and S is considered as an operator from Φ to Φ' .

IV.3 Scattering Amplitudes

We have assumed (see the beginning of Section 2) that the S-matrix S is unitary. This implies that it is continuous in \mathcal{H}_s and, therefore, can be identified as an element of $L^\times(\Phi; \Phi')$. As such, it has a corresponding kernel $\langle S \rangle$ (more precisely, each element $S^{(m,n)} \in L(\mathcal{H}^{(n)}; \mathcal{H}^{(m)})$, as defined in Section III.3, has a corresponding kernel $\langle S^{(m,n)} \rangle \in \mathcal{D}(\Omega^m \times \Omega^n)$). The momentum operator P is a generator of \mathcal{P} . Applying eq. (IV.15) we get

$$[P\varphi](p_1, \dots, p_n) = (p_1 + \dots + p_n)\varphi(p_1, \dots, p_n), \quad \forall \varphi \in \bigotimes_1^n \Phi^{(1)} \quad (\text{IV.20})$$

and applying eq. (IV.18), we get, for all $\varphi \in \bigotimes_1^n \Phi^{(1)}$,

$$0 = ([P, S]\varphi)(q^m) = ([P, S^{(m,n)}]\varphi)(q^m) = \quad (\text{IV.21})$$

$$= \int d\mu^n(p^n) \left(\sum_1^m q_j - \sum_1^n p_i \right) \langle q^m | S | p^n \rangle \varphi(p^n) d\mu^n(p^n), \quad (\text{IV.22})$$

and hence

$$\left(\sum_1^m q_j - \sum_1^n p_i \right) \langle q^m | S | p^n \rangle = 0; \quad (\text{IV.23})$$

obviously this holds also for $S - I$ replacing S , where I is the identity operator in \mathcal{H} . This implies that $\langle S - I \rangle$ is of the form [20, vol. 1]

$$\langle S - I \rangle = -i(2\pi)^d \delta^d \left(\sum_1^m q_j - \sum_1^n p_i \right) \langle T \rangle \quad (\text{IV.24})$$

where $d = r + 1$ is the dimension of the momentum space, $-i(2\pi)^d$ is a conventional normalization factor and $\langle T \rangle$ is a generalized function on the submanifold of $\Omega^m \times \Omega^n$ defined by the constraint

$$\sum_1^m q_j - \sum_1^n p_i = 0. \quad (\text{IV.25})$$

(this is the precise formulation of energy-momentum conservation) so finally:

$$\langle S^{(m,n)} \rangle = \langle I^{(m,n)} \rangle - i(2\pi)^d \delta^d \left(\sum_1^m q_j - \sum_1^n p_i \right) \langle T^{(m,n)} \rangle. \quad (\text{IV.26})$$

The (generalized) values of $\langle T \rangle$ are called “scattering amplitudes”.

IV.4 The Optical Theorem

Since the indices $[\alpha\lambda]$ are omitted, $\langle q^m | T | p^n \rangle$ is a (matrix) operator from $\mathcal{H}(p^n) := \bigotimes_1^n \mathcal{H}(p_i)$ to $\mathcal{H}(q^m)$ (defined similarly) and the integration $d\mu^n(p^n)$ is actually over $\hat{\mathcal{T}}_F^n \equiv \prod_1^n \hat{\mathcal{T}}_F$. In this Section, “ \dagger ” denotes the Hermitian conjugation of matrices. Bearing all this in mind, we now show that the unitarity of S leads to

The Optical Theorem:

$$\begin{aligned} \langle p^n | T | p^n \rangle - \langle p^n | T | p^n \rangle^\dagger = \\ i \sum_{m=0}^{\infty} \int_{\hat{\mathcal{T}}_F^m} d\mu^m(q^m) (2\pi)^d \delta^d \left(\sum_1^m q_j - \sum_1^n p_i \right) \langle p^n | T | q^m \rangle^\dagger \langle q^m | T | p^n \rangle \end{aligned} \quad (\text{IV.27})$$

Proof:

The unitarity of S implies that:

$$(I - S)^*(I - S) = (I - S) + (I - S)^*. \quad (\text{IV.28})$$

Writing the corresponding equation for kernels (using the expression (IV.24) for $\langle I - S \rangle$) we get

$$\begin{aligned} 0 &= \langle p^n | [(I - S) + (I - S)^* - (I - S)^*(I - S)] | r^l \rangle = \\ &= i(2\pi)^d \delta^d \left(\sum_1^n p_i - \sum_1^l r_k \right) \times \\ &\quad \times \{ \langle p^n | T | r^l \rangle - \langle p^n | T | r^l \rangle^\dagger - \sum_{m=1}^{\infty} \int_{\hat{\mathcal{T}}_F^m} d\mu^m(q^m) \\ &\quad i(2\pi)^d \delta^d \left(\sum_1^m q_j - \sum_1^n p_i \right) \langle p^n | T | q^m \rangle^\dagger \langle q^m | T | r^l \rangle \} \end{aligned} \quad (\text{IV.29})$$

so for $\sum_1^n p_i = \sum_1^l r_k$, the expression in $\{\}$ must vanish. In particular, for $r^l = p^n$. ■

V. Comments and Supplements

In this Section we discuss the assumptions used for the construction, emphasizing the prospects for relaxing some of them.

V.1 Other Signatures and Orbits

In Section 3 we assumed signatures of the type $(r, 1)$ and representations with momentum support in the forward light cone. All this was needed to assure the compactness of the

little group, which implies that its irreducible representation spaces $\{\mathcal{H}(L^\alpha)\}$ are finite-dimensional. This plays a key role in the construction of the Gel'fand triple. If $\mathcal{H}(L^\alpha)$ is of infinite dimension, it is never nuclear [17, p. 520] and L^α is, in general, not smooth. Therefore the choice $\Psi_\alpha := \{\mathcal{H}(L^\alpha)\}$ made in Section III.1 cannot satisfy in this case the requirements of Section II.2 for the construction of a Gel'fand triple suitable for $\mathcal{A}_\mathcal{P}$. If L^α itself is induced by a finite dimensional representation, the space Ψ_α can be constructed by the same procedure described in Section 2.3 for Φ_α . Using this procedure, Φ_α can be constructed for any representation which can be built by a sequence of inductions, starting with a finite-dimensional representation.

The infinite dimension of $\{\mathcal{H}(L^\alpha)\}$ may cause another complication. In this case, the spectrum σ_α (of the operator J used to represent $\{\mathcal{H}(L^\alpha)\}$ as a space of functions – eq. (II.12)) is not necessarily discrete. If it is continuous, Ω (defined in eq. (II.17)) is not a countable union of orbits so to consider Ω as a smooth separable manifold, one must include a differential structure on σ_α ; this must be taken into account when checking the smoothness of functions on Ω . If the spectrum is mixed, Ω is a union of manifolds of different dimension.

Finally, the choice of signature $(r, 1)$ and momenta in the forward light cone has also a physical significance. In this region p^0 is bounded from below (positive), thus suitable to be interpreted as the energy. In any other case (except for the forward light-like momenta in the case of signature $(r, 1)$) the orbits are unbounded in all directions, and therefore the canonical energy is not well defined (recall that the energy is distinguished from other components of the momentum by being positive and this in an invariant — and therefore well defined — statement only in the case of signature $(r, 1)$).

V.2 The Particle-Type Spectrum

A particle type is identified with an irreducible representation of \mathcal{P} . The assumption that it is a *true* representation (assumption 2.2 of Section II.1) is used in the extension of the representation U from \mathcal{P} to its algebra $\mathcal{A}_\mathcal{P}$ (see appendix B.2). When U is a genuine projective representation (with a non-trivial phase), \tilde{U} (defined by eq. (B.23)) is not a representation of $\mathcal{E}'(\mathcal{P})$ (it doesn't conserve multiplication).

The discreteness of the set of particle types (assumption 3.2 of Section 3) was used (with the finiteness of $\dim \mathcal{H}(L^\alpha)$) to identify Ω as a countable union of orbits, as discussed in the previous subsection.

VI. Conclusions

In this work we constructed explicitly and rigorously a basis of “plane-waves” – momentum (generalized) eigenstates – for the space of states used to describe relativistic scattering. We exploited the assumed \mathcal{P}_0 symmetry, and used the theory of induced representations and the structure of Gel'fand triples. To combine rigor and clarity we used the a rigorized version of Dirac's “bra” and “ket” formalism. We develop this formalism

further and introduce a convenient notation which distinguishes bra's $\langle \cdot |, | \cdot \rangle$ from ket's $(\cdot |, | \cdot)$ [21]. This notation made it possible to use the “complete set of states” to decompose expressions into “vector components” and “matrix elements” in almost the same flexibility as in the original formalism. We demonstrate this flexibility in a few examples. A further demonstration is given in [7], where the construction of the present work is used to prove an extension of the Coleman-Mandula theorem.

Appendix A. The Dirac Formalism

A.1 Conventional Terminology

A.1.1 Spaces of Operators

(In this subsection, E, F are topological vector spaces over the complex field)

- $L(E, F)$: the space of continuous linear mappings from E to F ;
- $L^\times(E, F)$: the space of continuous antilinear mappings from E to F ;
($L(E, F)$ and $L^\times(E, F)$ are naturally [antilinearly] isomorphic)
- $L(E) := L(E, E)$ and $L^\times(E) := L^\times(E, E)$;
- $E' := L(E; \mathbf{C})$: “the dual of E ”; the space of continuous linear functionals on E ;
when endowed with the “strong dual topology”, it is called “the strong dual”;
- $\bar{E}' := L^\times(E, \mathbf{C})$.

When E is reflexive (which means that it is the strong dual of its strong dual) then $(\bar{E}')'$ is naturally (antilinearly) isomorphic to E and therefore is denoted by \bar{E} .

A.1.2 Spaces of Functions

(In this subsection, Ω is a separable smooth (differentiable) manifold, μ is a measure on Ω and $\mathcal{H}, \mathcal{H}(x)$ are Hilbert spaces)

- $\int_\Omega^\oplus d\mu(x) \mathcal{H}(x)$: a direct integral of Hilbert spaces. An element of this space is a vector field

$$f : x \in \Omega \mapsto f(x) \in \mathcal{H}(x).$$

This is a Hilbert space with respect to the inner product

$$(f, g) := \int_\Omega d\mu(x) (f(x), g(x))_{\mathcal{H}(x)}$$

where $(\cdot, \cdot)_{\mathcal{H}(x)}$ is the inner product in $\mathcal{H}(x)$;

- $\mathcal{L}_\mu^2(\Omega; \mathcal{H}) := \int_\Omega^\oplus d\mu(x) \mathcal{H}$: the space of μ -square-integrable functions from Ω to \mathcal{H} ;
- $\mathcal{E}(\Omega; \mathcal{H})$: the space $C^\infty(\Omega; \mathcal{H})$ of smooth (infinitely differentiable) functions on Ω , with values in \mathcal{H} , equipped with the “Schwartz topology” (uniform convergence on every compact set in Ω of the functions and all their derivatives);
- $\mathcal{D}(\Omega; \mathcal{H})$: the space $C_c^\infty(\Omega; \mathcal{H})$ of those elements of $C^\infty(\Omega; \mathcal{H})$ that have compact support, equipped with the “Schwartz topology”. (A sequence of functions $\varphi_k \in \Phi$ converges in this topology iff they have a common compact subset of Ω containing their supports and for each differential operator D , the sequence $\{D\varphi_k\}$ converges uniformly [22, p. 147]). The elements of $\mathcal{D}(\Omega; \mathcal{H})$ are called (\mathcal{H} -valued) “test functions on Ω ”.

These topologies were introduced by Schwartz [23] and are nuclear [15, p. 69] and complete [17]. $\mathcal{D}(\Omega; \mathcal{H})$ is also reflexive [17];

- $\mathcal{D}'(\Omega; \mathcal{H})$: the strong dual of $\mathcal{D}(\Omega; \mathcal{H})$. Its elements are called (\mathcal{H} -valued) “distributions on Ω ”;
- $\mathcal{E}'(\Omega; \mathcal{H})$: the strong dual of $\mathcal{E}(\Omega; \mathcal{H})$. Consists of those elements of $\mathcal{D}'(\Omega; \mathcal{H})$ that have compact support.

(When $\mathcal{H} = \mathbf{C}$, the field of complex numbers, this label is omitted. *e.g.* $\mathcal{L}_\mu^2(\Omega, \mathbf{C}) = \mathcal{L}_\mu^2(\Omega)$.)

A.2 The Space of States

The space of states is a Gel’fand triple (originally called “rigged Hilbert space” by Gel’fand *et al.* [20]) – a triplet $(\Phi, \mathcal{H}, \Phi')$ of topological vector spaces with the following properties:

1. \mathcal{H} is a complex separable Hilbert space;
2. Φ is a dense subspace of \mathcal{H} , equipped with a finer topology (more open sets; this is equivalent to the statement that the embedding of Φ in \mathcal{H} is continuous);
3. Φ' is the strong dual of Φ (*i.e.* the topological dual, equipped with the strong dual topology [17]).

In the present context, it is further required that Φ be complete, “nuclear” [15, 17] and reflexive (Φ is the strong dual of Φ').

The space of states is equipped with a “complete set of commuting observables” $\{A_i\}$: mutually strongly commuting self adjoint operators which, when restricted to Φ are continuous in the topology of Φ [24]. According to the complete spectral theorem of von Neumann [15, p.54], \mathcal{H} is isomorphic to $\mathcal{L}_\mu^2(\Omega)$, where Ω is the combined spectrum of $\{A_i\}$. The observables are so chosen that Ω is a separable differentiable manifold (or a discrete union of such manifolds), μ is a non degenerate Radon measure on Ω and $\Phi = \mathcal{D}(\Omega)$.

A.3 The Elements in the Formalism

The types of vectors in $(\Phi, \mathcal{H}, \Phi')$:

- **ket vectors:** elements $|\varphi\rangle, |\psi\rangle, \dots$ of Φ and elements $\langle\varphi|, \langle\psi|, \dots$ of $\overline{\Phi}$;
- **normalizable vectors:** elements f, g, \dots of \mathcal{H} ;
- **bra vectors:** elements $\langle\xi|, \langle\zeta|, \dots$ of Φ' and elements $|\xi\rangle, |\zeta\rangle, \dots$ of $\overline{\Phi'}$.

There are three products

- $(f, g)_{\mathcal{H}}$ is the inner product (linear in g) in \mathcal{H} (the subscript \mathcal{H} is usually omitted);
- $\langle\xi|\varphi\rangle$ is the dual action between $\langle\xi|$ and $|\varphi\rangle$ (the Dirac “bracket”);
- $\langle\varphi|\xi\rangle$ is the dual action between $\langle\varphi|$ and $|\xi\rangle$ (this would deserve the name “ketbra”...).

Note that by definition

$$\langle\varphi|\xi\rangle = \overline{\langle\xi|\varphi\rangle}. \quad (\text{A.1})$$

The operators are elements of $L^\times(\Phi; \Phi')$, a space containing also the spaces $L(\Phi)$, $L(\mathcal{H})$, $L(\Phi')$, $L(\Phi; \mathcal{H})$ and $L^\times(\mathcal{H}; \Phi')$ (after identifying \mathcal{H} and Φ as subspaces of Φ' and restricting mappings from \mathcal{H} and Φ' to Φ).

A.4 Definitions

The map $f \mapsto f'$ defined by

$$\langle f'|\varphi\rangle := (f, \varphi), \quad \forall \varphi \in \Phi \quad (\text{A.2})$$

is the natural (antilinear) embedding of \mathcal{H} as a (sequentially) dense subspace of Φ' . (The prime in f' is usually dropped.)

A^* denotes the Hilbert-space-adjoint of an operator A in \mathcal{H} .

$A^\dagger \in L^\times(\Phi; \Phi')$, “the adjoint” of $A \in L^\times(\Phi; \Phi')$, is defined by

$$\langle A^\dagger \psi|\varphi\rangle := \langle\psi|A\varphi\rangle, \quad \forall \varphi, \psi \in \Phi. \quad (\text{A.3})$$

If $A(\Phi) \subset \mathcal{H}$ (so that A^* is uniquely defined), then for each $\varphi \in \Phi$

$$A^*\varphi \text{ is defined } \iff A^\dagger\varphi \in \mathcal{H} \quad (\text{A.4})$$

and in this case $A^*\varphi = A^\dagger\varphi$, so A^\dagger is the extension to all of Φ' of the restriction of A^* to Φ .

$B' \in L(\Phi')$, “the dual” of $B \in L(\Phi)$, is defined by

$$\langle B'\xi|\varphi\rangle := \langle\xi|B\varphi\rangle, \quad \forall \varphi \in \Phi, \xi \in \Phi' \quad (\text{A.5})$$

and satisfies, for each $\psi \in \Phi$

$$B'\psi' = B^\dagger\psi \text{ (} = (B^*\psi)', \text{ when defined)} \quad (\text{A.6})$$

so A' extends A^\dagger (and thus also A^*) from Φ to Φ' .

The following definitions are made: (for $B \in L(\Phi), \varphi, \psi \in \Phi, \xi \in \Phi'$)

$$\langle \xi | B | \varphi \rangle := \langle \xi | B \varphi \rangle (= \langle B' \xi | \varphi \rangle) \quad (\text{A.7})$$

$$(\psi | B | \varphi) := \langle \psi' | B | \varphi \rangle = (\psi, B \varphi) \quad (\text{A.8})$$

and when $B^*\psi$ is defined, also

$$(\psi | B | \xi \rangle := \overline{\langle \xi | B^* | \psi \rangle} = (B^* \psi | \xi \rangle. \quad (\text{A.9})$$

The (generalized) basis of bras for Φ suggested naturally by this construction is:

$$\{\langle x | \mid x \in \Omega\} \quad (\text{A.10})$$

where (for μ -almost all $x \in \Omega$) $\langle x |$ is defined by

$$\langle x | \varphi \rangle := \varphi(x), \quad \forall \varphi \in \Phi. \quad (\text{A.11})$$

This is a “complete, orthonormal” [25] system of eigenvectors of A in the sense that the following relations are satisfied:

1. The Parseval equality:

$$(\psi, \varphi) = \int_{\Omega} d\mu(x) (\psi | x \rangle \langle x | \varphi), \quad \forall \varphi, \psi \in \Phi. \quad (\text{A.12})$$

2. The eigenvalue equation:

$$\langle x | A_i \varphi \rangle = x_i \langle x | \varphi \rangle, \quad \forall \varphi \in \Phi, x \in \Omega \quad (\text{A.13})$$

or more briefly

$$A'_i \xi_x = x_i \xi_x \quad (\text{A.14})$$

Generalizing (A.11), we denote (for $\xi \in \Phi'$)

$$\langle \xi | x \rangle \equiv \xi(x), \quad \langle x | \xi \rangle \equiv \bar{\xi}(x), \quad (\text{A.15})$$

the generalized “values” of the generalized functions ξ and $\bar{\xi}$. ($\bar{\xi}$ is defined by $\langle \bar{\xi} | \varphi \rangle := \overline{\langle \xi | \bar{\varphi} \rangle}$.)

Finally, the matrix elements $\langle x | A | y \rangle$ of an operator $A \in L^\times(\Phi; \Phi')$ are defined to be the generalized “values” of the kernel $\langle A \rangle \in \mathcal{D}'(\Omega \times \Omega)$ satisfying

$$(\psi | A | \varphi) = \int_{\Omega} d\mu(x) \int_{\Omega} d\mu(y) (\psi | x \rangle \langle x | A | y \rangle \langle y | \varphi). \quad (\text{A.16})$$

(Such a kernel does exist, by “the kernel theorem” of Schwartz, which states [17, p. 531] that there is a natural isomorphism between $L^\times(\Phi; \Phi')$ and $\mathcal{D}'(\Omega \times \Omega)$.)

A.5 The Rules

All the above definitions obey the following three rules:

1. Whenever two of the expressions $(f|g >, < f|g)$ and (f, g) are defined (*i.e.* when $f, g \in \mathcal{H}$ and at least one of them is in Φ), they are equal:

$$(f|g > = < f|g) = (f, g) = \int_{\Omega} d\mu(x) \overline{f(x)} g(x). \quad (\text{A.17})$$

The same is true for $(f|B|g >, < f|B|g)$ and (f, Bg) .

2. The conjugate of a bracket product (*e.g.* $< \quad | \quad)$ or $(\quad | \quad)$) is the product of the conjugates in reverse order (if this last product is defined), where under conjugation \dots :

$$\begin{aligned} \varphi \in \Phi & : (\varphi| \longleftrightarrow |\varphi) \\ \xi \in \Phi' & : < \xi| \longleftrightarrow |\xi > \\ A \in L^\times(\Phi; \Phi') & : A \longleftrightarrow A^\dagger \end{aligned} \quad (\text{A.18})$$

3. In any well defined expression of the form $(\psi|A_1 \dots A_k|\varphi)$ or $< x|A_1 \dots A_k|\varphi)$ or $< x|A_1 \dots A_k|y >$, the expression

$$I := \int_{\Omega} d\mu(x) |x > < x| \quad (\text{A.19})$$

can be formally inserted between any two factors to obtain a decomposition in terms of components and matrix elements (note that the Parseval equality (A.12) is the simplest example of this rule). Mathematically, I is the embedding of Φ into $\overline{\Phi'}$ and of $\overline{\Phi}$ into Φ' .

This is very close to the original rules introduced by Dirac, but there are some complications:

- The conjugate of a bra $< \xi|$ is $|\xi >$, which is also a bra and in general not a ket (since not every bra has a corresponding ket) so the conjugate of a ket $|\varphi)$ should be seen as a ket $(\varphi|$ (this is why the kets and the bras are denoted by different symbols).
- The conjugate of an operator $A \in L^\times(\Phi; \Phi')$ is the adjoint A^\dagger . This works fine between kets but between a bra and a ket (when $A \in L(\phi)$), for the conjugate expression to be defined, the Hilbert-space-adjoint A^* must be defined (see eq. (A.9)).
- In the original Dirac formalism I is the identity operator and not an embedding, therefore this formalism is not recovered here fully; not every expression allowed by the formalism is well defined.

Appendix B. The Group $\mathcal{P}(r, s)$

$\mathcal{O}(r, s)$, called “the pseudo-orthogonal group of signature (r, s) ”, is the group of all linear transformations in \mathbf{R}^{r+s}

$$x \mapsto x' = \Lambda x \quad (\text{B.1})$$

(where Λ is a real $(r+s) \times (r+s)$ matrix) that conserves the quadratic form $x \cdot x$, where

$$x \cdot y \equiv x_\mu y^\mu \equiv \eta_{\mu\nu} x^\mu y^\nu \quad , \quad \eta = \text{diag}(\overbrace{1, \dots, 1}^s, \overbrace{-1, \dots, -1}^r). \quad (\text{B.2})$$

$\mathcal{P}(s, r)$, called “the inhomogeneous pseudo-orthogonal group”, is the group of all affine transformations

$$x \mapsto x' = \Lambda x + a, \text{ where } \Lambda \in \mathcal{O}(r, s), a \in \mathbf{R}^{r+s}. \quad (\text{B.3})$$

The composition law in $\mathcal{P}(r, s)$ is (according to eq. (B.3))

$$(\Lambda_2, a_2)(\Lambda_1, a_1) = (\Lambda_2 \Lambda_1, \Lambda_2 a_1 + a_2) \quad (\text{B.4})$$

and the inverse is

$$(\Lambda, a)^{-1} = (\Lambda^{-1}, -\Lambda^{-1}a) \quad (\text{B.5})$$

thus $\mathcal{P}(r, s)$ is the semidirect product of \mathcal{T}_{r+s} , the translation group in \mathbf{R}^{r+s} , and of $\mathcal{O}(r, s)$.

$\mathcal{O}_0(r, s)$ and $\mathcal{P}_0(r, s)$ denote the identity component (the largest connected subgroup) of $\mathcal{O}(r, s)$ and $\mathcal{P}(r, s)$ respectively and $\overline{\mathcal{O}}_0(r, s)$ and $\overline{\mathcal{P}}_0(r, s)$ denote their universal covering groups. To symplify notation, $\mathcal{P}_0(r, s)$, $\overline{\mathcal{P}}_0(r, s)$, $\mathcal{O}_0(r, s)$, $\overline{\mathcal{O}}_0(r, s)$ and \mathcal{T}_{r+s} will be abbreviated by \mathcal{P}_0 , \mathcal{P} , \mathcal{L}_0 , \mathcal{L} and \mathcal{T} respectively.

There is a homomorphism from \mathcal{L} onto \mathcal{L}_0 . Its kernel is [12] $\mathcal{N}(r) \otimes \mathcal{N}(s)$ where $\mathcal{N}(r)$ is a cyclic group:

$$\mathcal{N}(r) = \begin{cases} \mathbf{Z}_1 & r = 1 \\ \mathbf{Z} & r = 2 \\ \mathbf{Z}_2 & r \geq 3. \end{cases} \quad (\text{B.6})$$

\mathcal{T} is connected and simply connected, therefore

- \mathcal{P}_0 is a semidirect product of \mathcal{T} and \mathcal{L}_0 .
- \mathcal{P} is a semidirect product of \mathcal{T} and \mathcal{L} .
- The homomorphism from \mathcal{L} onto \mathcal{L}_0 extends naturally to a homomorphism from \mathcal{P} onto \mathcal{P}_0 , with the same kernel, therefore \mathcal{P}_0 is the quotient group

$$\mathcal{P}_0 = \mathcal{P} / (\mathcal{N}(r) \otimes \mathcal{N}(s)) \quad (\text{B.7})$$

The homomorphism from \mathcal{P} onto \mathcal{P}_0 identifies any (projective or true) representation of \mathcal{P}_0 as a (projective or true) representation of \mathcal{P} . Such a homomorphism, from \mathcal{P} to \mathcal{P}'_0 (also with an abelian discrete kernel), exists for any connected group \mathcal{P}'_0 which is locally isomorphic to \mathcal{P}_0 .

B.1 Induced Representations of \mathcal{P}

\mathcal{T} is an Abelian group (isomorphic to \mathbf{R}^{r+s}) and thus its dual $\hat{\mathcal{T}}$ – the set of non-equivalent irreducible continuous unitary representations of \mathcal{T} – consists of characters:

$$\hat{\mathcal{T}} = \{\chi_p : a \mapsto e^{ip \cdot a}, \forall a \in \mathcal{T} | p \in \mathbf{R}^{r+s}\}. \quad (\text{B.8})$$

$\hat{\mathcal{T}}$ is isomorphic to \mathbf{R}^{r+s} and will be called “the momentum space” (following the physicists’ terminology in the 4-D case). \mathcal{P} acts naturally on $\hat{\mathcal{T}}$:

$$(\Lambda, a) : p \mapsto \Lambda_0 p \quad (\text{B.9})$$

where $\Lambda_0 \in \mathcal{L}_0$ is the image of $\Lambda \in \mathcal{L}$ (notice that from now on Λ denotes an element of \mathcal{L} and not of \mathcal{L}_0 , although it corresponds to an element of \mathcal{L}_0). \mathcal{T} acts trivially, so we can say that only \mathcal{L} acts. This action will be denoted simply by Λp . It decomposes $\hat{\mathcal{T}}$ into orbits. These are classified according to the “stability group” $\mathcal{L}(p) = \{\Lambda \in \mathcal{L} | \Lambda p = p\}$ of their elements (called also “isotropy group” or “little group”). To classify the orbits, we denote [26]:

p_+ the vector of the first s components of p ,

p_- the vector of the last r components of p .

If $r, s > 1$ the orbits are characterized by $p_\mu p^\mu = p_+^2 - p_-^2$ ($p_\mu p^\mu = 0$ splits to two orbits: $p = 0$ and $p \neq 0$). If $s = 1$, some of the orbits split to two: $p_+ > 0$ and $p_+ < 0$ and similarly for $r = 1$. Table B.1 lists the orbits, a representative p_0 of each orbit and the little group of p_0 .

Given an orbit $\hat{\mathcal{T}}_m$ (for simplicity, unless otherwise stated, the subscript m is here an abstract symbol of an arbitrary type of orbit and not necessarily the time-like mass $m = \sqrt{p_\mu p^\mu}$), a representative $p_m \in \hat{\mathcal{T}}_m$ and a unitary representation L of $\mathcal{L}(p_m)$ in a (complex separable) Hilbert space $\mathcal{H}(L)$, L induces a representation U of \mathcal{P} in $\mathcal{L}_{\mu_m}^2(\hat{\mathcal{T}}_m, \mathcal{H}(L))$, where μ_m is the non trivial \mathcal{L}_0 -invariant Radon measure on $\hat{\mathcal{T}}_m$ (which exists, since both \mathcal{L}_0 and $\mathcal{L}_0(p_m)$ are unimodular, and is unique up to a multiplicative constant). First, one chooses for each $p \in \hat{\mathcal{T}}_m$, an element Λ_p of \mathcal{L} that satisfies $\Lambda_p p_m = p$, and then U is defined by:

$$[U(\Lambda, a)f](p) := e^{ip \cdot a} L(\Delta(\Lambda, p)) f(\Lambda^{-1}p) \quad (\text{B.10})$$

$$\text{where } \Delta(\Lambda, p) := \Lambda_p^{-1} \Lambda \Lambda_{\Lambda^{-1}p} \in \mathcal{L}(p_m).$$

For convenience, we choose $\Lambda_{p_m} = 1$ to obtain

$$[U(\Lambda_p)f](p) = f(p_m), \quad \forall p \in \hat{\mathcal{T}}_m \quad (\text{B.11})$$

$$[U(\Delta)f](p_m) = L(\Delta)f(p_m) \quad \forall \Delta \in \mathcal{L}(p_m). \quad (\text{B.12})$$

U is denoted by

$$U = (\chi_{p_m} L)(\mathcal{P}(p_m)) \upharpoonright \mathcal{P} \quad (\text{B.13})$$

Table B.1: The orbits of $\mathcal{L}_0 = \mathcal{O}_0(r, s)$

indicating that it is a representation of all of \mathcal{P} “lifted” from a representation $\chi_{p_m} L$ of a subgroup $\mathcal{P}(p_m)$ of \mathcal{P} . This is a unitary representation and if L and the mapping $p \mapsto \Lambda_p$ are continuous then so is U .

The set $\{p_m\}$ of representatives of the orbits (as chosen in the table) is obviously a measurable set in $\hat{\mathcal{T}}$ so, according to Mackey’s theorem [15, p. 279], every unitary representation of \mathcal{P} is equivalent to an induced representation and it is irreducible iff the inducing representation (the one on the little group) is irreducible. Thus an irreducible unitary representation on \mathcal{P} is characterized by an orbit and an irreducible unitary representation of the little group of this orbit. Combining this with the theory of decomposition of a continuous unitary representation U [15, p. 162] and with the assumption that the factors of U are all of type I, one obtains that the most general form of U is as described in Section 2.2.

B.2 Extending the Representation to Generators

We denote:

$\mathcal{A}_{\mathcal{P}}$: the Lie algebra of \mathcal{P} (generators of \mathcal{P}),

$\mathcal{U}_{\mathcal{P}}$: the universal enveloping algebra of $\mathcal{A}_{\mathcal{P}}$ (polynomials in elements of $\mathcal{A}_{\mathcal{P}}$).

\mathcal{P} is a separable smooth manifold so the spaces $\mathcal{E}(\mathcal{P})$ and $\mathcal{E}'(\mathcal{P})$ (see Section A.1.2) are well defined. $\mathcal{E}'(\mathcal{P})$ is a $*$ algebra with respect to

multiplication (a “convolution”): for each $T_1, T_2 \in \mathcal{E}'(\mathcal{P})$, $T_1 * T_2$ is defined by

$$\int_{\mathcal{P}} (T_1 * T_2)(x) f(x) dx := \int_{\mathcal{P} \times \mathcal{P}} T_1(x) T_2(y) f(xy) dx dy, \quad \forall f \in \mathcal{E}(\mathcal{P}) \quad (\text{B.14})$$

and

involution (“conjugation”): The conjugate T^\dagger of $T \in \mathcal{E}'(\mathcal{P})$ is defined by

$$\int T^\dagger(x) f(x) dx := \overline{\int T(x) f^\dagger(x) dx}, \quad \forall f \in \mathcal{E}(\mathcal{P}) \quad (\text{B.15})$$

where $f^\dagger(x) := \overline{f(x^{-1})}$.

Now $\mathcal{A}_{\mathcal{P}}$ is, by definition, the tangent space to \mathcal{P} at the identity e . Thus, in a coordinate neighborhood (x_1, \dots, x_n) of the identity $x = 0$, $A \in \mathcal{A}_{\mathcal{P}}$ is expressed by:

$$Af = a_i \frac{\partial}{\partial x_i} f(x)|_{x=0} = - \int (a_i \frac{\partial}{\partial x_i} \delta(x)) f(x) dx, \quad \forall f \in \mathcal{E}(\mathcal{P}) \quad (\text{B.16})$$

so A acts in $\mathcal{E}(\mathcal{P})$ as a distribution: a linear combination of derivatives of δ_e (the δ -function with support in the identity e of \mathcal{P} : $\delta_e f = f(e)$). This identifies $\mathcal{U}_{\mathcal{P}}$ naturally with a sub- $*$ -algebra of $\mathcal{E}'(\mathcal{P})$:

$$\mathcal{U}_{\mathcal{P}} \simeq \mathcal{E}'_e(\mathcal{P}) := \{T \in \mathcal{E}'(\mathcal{P}) | \text{supp}(T) \subset \{e\}\} \quad (\text{B.17})$$

which is spanned by derivatives of δ_e , while $\mathcal{A}_{\mathcal{P}}$ is the subspace of first order derivatives. We denote by $T_A \in \mathcal{E}'(\mathcal{P})$ the distribution corresponding to $A \in \mathcal{U}_{\mathcal{P}}$ and we get:

$$T_{AB} = T_A * T_B \quad \forall A, B \in \mathcal{U}_{\mathcal{P}} \quad (\text{B.18})$$

$$T_{A^\dagger} = T_A^\dagger \quad \forall A \in \mathcal{U}_{\mathcal{P}} \quad (\text{B.19})$$

where the involution of $A \in \mathcal{A}_{\mathcal{P}}$ is defined by $A^\dagger := -A$ and extended to $\mathcal{U}_{\mathcal{P}}$ by $(AB)^\dagger := B^\dagger A^\dagger$. This means that the map $A \mapsto T_A$ is a $*$ isomorphism from $\mathcal{U}_{\mathcal{P}}$ onto $\mathcal{E}'(\mathcal{P})$.

\mathcal{P} is also embedded naturally in $\mathcal{E}(\mathcal{P})$ by the map $x \mapsto T_x := \delta_x$ in the following sense:

1. $T_{xy} = T_x * T_y$, $\forall x, y \in \mathcal{P}$ so $x \mapsto T_x$ is an isomorphism,
2. if e^{tA} is a one-parameter subgroup of \mathcal{P} , generated by $A \in \mathcal{A}_{\mathcal{P}}$ then

$$T_A f = \frac{d}{dt} f(e^{tA})|_{t=0} = \left[\int \frac{d}{dt} \delta_{e^{tA}}(x) f(x) dx \right]_{t=0} \quad (\text{B.20})$$

and this means that

$$\frac{d}{dt} T_{e^{tA}}|_{t=0} = T_A = T_{\frac{d}{dt}(e^{tA})|_{t=0}}. \quad (\text{B.21})$$

Now let L be an irreducible unitary representation of $\mathcal{L}(p_m)$ in $\mathcal{H}(L)$, inducing a representation U on $\mathcal{H} = \mathcal{L}_{\mu_m}^2(\hat{\mathcal{T}}_m; \mathcal{H}(L))$, as defined in Section B.1, and suppose:

1. $(\Psi, \mathcal{H}(L), \Psi')$ is a Gel'fand triple: Ψ is a complete nuclear space, embedded in $\mathcal{H}(L)$ densely and continuously,
2. the restriction of L to Ψ is a smooth representation of $\mathcal{L}(p_m)$ by continuous operators in Ψ :
 - (a) Ψ is L -invariant,
 - (b) for each $\Delta \in \mathcal{L}(p_m)$, $L(\Delta)$ is a continuous operator in Ψ ,
 - (c) for each $\psi \in \Psi$, the function $\Delta \mapsto L(\Delta)\psi$ is a smooth function from $\mathcal{L}(p_m)$ to $\mathcal{H}(L)$ (an element of $\mathcal{E}(\mathcal{L}(p_m); \mathcal{H}(L))$),
3. the map $p \mapsto \Lambda_p$ is smooth on $\hat{\mathcal{T}}_m$ (this can always be satisfied).

Defining

$$\Phi := \mathcal{D}(\hat{\mathcal{T}}_m; \Psi), \quad (\text{B.22})$$

we get:

- $(\Phi, \mathcal{H}, \Phi')$ is a Gel'fand triple with all the properties in (1),
- the restriction of U to Φ is a smooth representation of \mathcal{P} by continuous operators in Φ .

The smoothness of U in Φ allows the definition:

$$\tilde{U}(T)\varphi := \int_{\mathcal{P}} d(\Lambda, a)T(\Lambda, a)U(\Lambda, a)\varphi, \quad \forall T \in \mathcal{E}'(\mathcal{P}), \varphi \in \Phi. \quad (\text{B.23})$$

Observing that

$$U(\Lambda, a) = \tilde{U}(\delta_{(\Lambda, a)}), \quad (\text{B.24})$$

we see that the natural extension of U to $\mathcal{U}_{\mathcal{P}}$ is

$$U(A) := \tilde{U}(T_A) \quad (\text{B.25})$$

and it can be shown [5, p. 2285] that U has the properties 3, 4 and 5, enumerated in Section 2.2.

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This approach has become standard in Quantum Field Theory; see for instance
N. N. Bogolubov, A. A. Logunov and I. T. Todorov, *Introduction to Axiomatic Quantum Field Theory*, Benjamin, New York 1975.
- [7] L.P. Horwitz and O. Pelc, *Generalization of the Coleman-Mandula Theorem to Higher Dimension*, RI-2-96, TAUP 2175-94 preprint.
- [8] A. O. Barut and R. Rączka, *Theory of Group Representations and Applications*, PWN, Warszawa 1977.

- [9] All the properties of \mathcal{P} used in this article (mainly its relation to \mathcal{P}_0) are invariant under isomorphism, so we take \mathcal{P} to be the universal covering group of \mathcal{P}_0 (and not only isomorphic to it).
- [10] E. P. Wigner, Ann. Math. **40** (1939) 149; *Group Theory*, Academic Press Inc., N.Y. 1959.
- [11] V. Bargmann, J. Math. Phys. **5** (1964) 862.
- [12] Bargmann showed (V. Bargmann, Ann. Math. **59** (1954) 1) that for $r + s \geq 3$, all the ray representations of \mathcal{P} are (equivalent to) true representations, so in this case assumption 2.2 is not a restriction).
- [13] Assumption 2.3 means that $U^{(1)}$ is expressible in terms of irreducible representations and since these are identified with particle types (see the discussion in the introduction of [7]), this requirement is actually part of the physical interpretation.
- [14] It can be shown [8, p. 328] that the Nelson-Stinespring criterion is satisfied by:
 - (a) each element of $\mathcal{A}_{\mathcal{P}}$,
 - (b) each central element of $\mathcal{U}_{\mathcal{P}}$,
 - (c) each element of $\mathcal{U}_{\mathcal{K}}$, where \mathcal{K} is a compact and/or abelian subgroup of \mathcal{P} (this includes (a)),
 so result R4 assures the existence of many essentially self adjoint operators among the representatives of $\mathcal{U}_{\mathcal{P}}$.
- [15] K. Maurin, *General Eigenfunction Expansions and Unitary Representations of Topological Groups*, PWN, Warszawa 1968.
- [16] The bar denotes the Hilbert space completion, which in this case means the closure in $\mathcal{L}_{\mu^n}^2(\Omega^n)$, since it is a complete space. The second equality in (II.18) expresses the fact that $\bigotimes_1^n \mathcal{L}_{\mu}^2(\Omega)$ is dense in $\mathcal{L}_{\mu^n}^2(\Omega^n)$.
- [17] F. Trèves, *Topological Vector Spaces, Distributions and Kernels*, Academic Press Inc. New York, 1967.
- [18] Eq. (III.14) shows this for $\bigotimes_1^n \mathcal{D}(\Omega)$, but this is a dense subspace of $\mathcal{D}(\Omega^n)$ so the continuity of the bras extends (III.14) to all of $\mathcal{D}(\Omega^n)$. In any case, because of the density of $\bigotimes_1^n \mathcal{D}(\Omega)$ in $\mathcal{D}(\Omega^n)$, usually only elements of $\bigotimes_1^n \mathcal{D}(\Omega)$ are considered.
- [19] According to Stone's theorem (see M. Reed and B. Simon, *Methods of Modern Mathematical Physics*, volume I: *Functional Analysis*, Academic Press, N.Y. 1980, p. 265),

if the function $t \mapsto (f, U^{(1)}(g(t))h)$ is measurable for each $f, h \in \mathcal{H}^{(1)}$ (and particularly if $t \mapsto g(t)$ is continuous) then there exists a self adjoint operator A_g^0 in $\mathcal{H}^{(1)}$ that satisfies $U^{(1)}(g(t)) = e^{itA_g^0}$, which implies that

$$\frac{d}{dt}U^{(1)}(g(t))h|_{t=0} = iA_g^0h \quad (\text{B.26})$$

(meaning that if for some $h \in \mathcal{H}^{(1)}$, one of the sides exists then both sides exist and they are equal). Comparing this to the definition of A_g , one sees that for each $\varphi \in \Phi^{(1)}$:

$$A_g^0\varphi \text{ is defined} \iff A_g\varphi \in \mathcal{H}^{(1)}$$

and in this case $A_g^0\varphi = A_g\varphi$ (we obtained such relation between A^* and A^\dagger in appendix A.4).

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- [21] J. M. Jauch and J.-P. Marchand (Helv. Phys. Act. **39** (1966) 325) seem to be the first to use the notation “ $< \cdot | \cdot \rangle$ ” but this was only for $\varphi(x) = \langle x | \varphi \rangle$ and they did not consider $\langle x |$ as a distinct entity.
- [22] M. Reed and B. Simon *Methods of Modern Mathematical Physics*, volume I: *Functional Analysis*, Academic Press, N.Y. 1980, p. 147.
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- [24] The notion of a “complete set of commuting observables” may be formulated also in a pure algebraic manner, see:
J.-P. Antoine, G. Epifanio and C. Trapani, Helv. Phys. Act. **56** (1983) 1175.
- [25] This is only in the sense of the Parseval equality, which is, in a Hilbert space, characteristic to a complete orthonormal system. This is not completeness in the usual sense since it is only in Φ , while the $\langle x |$'s are not, in general, in Φ . Orthogonality is not even defined since there is no inner product in Φ' .
- [26] When $s = 1$ i.e. $\mathcal{L}_0 = \mathcal{O}_0(r, 1)$, the standard notation for the components of the momenta is:

$$p = (E, \vec{p}) \quad , \quad E = p^0 (= p_+) \quad , \quad \vec{p} = (p^1, \dots, p^r) (= p_-).$$

Captions

Table B.1: The orbits of $\mathcal{L}_0 = \mathcal{O}_0(r, s)$